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On covering vertices of a graph by trees

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Abstract

The purpose of this paper is to initiate study of the following problem: Let G be a graph, and $k \geqslant 1$. Determine the minimum number s of trees $T_1, \ldots, T_s$, $\Delta(T_i) \leqslant k$, $i = 1, \ldots, s$, covering all vertices of G. We conjecture: Let G be a connected graph, and $k \geqslant 2$. Then the vertices of G can be covered by $s \leqslant \left\lceil \frac{n-\delta}{\delta(k-1)+1} \right\rceil$ edge-disjoint trees of maximum degree $\leqslant k$. As a support for the conjecture we prove the statement for some values of δ and k.

Keywords: Covering vertices; Path; Tree

1. Introduction

Let G be a graph. We use $\delta(G)$, $\Delta(G)$, $\alpha(G)$, and $\kappa(G)$ for its minimum degree, the maximum degree, the independence number, and the vertex connectivity, respectively. When the graph is clear from the context we will drop the name of the graph and write simply δ and Δ. Throughout the paper n will stand for the number of vertices of G.

There are several ways how to relax a property that a graph G possesses a hamiltonian path. The following two seem to be the most frequently studied. The first one is to see a hamiltonian path as a spanning tree of degree 2 and seek, in its absence, conditions under which G possesses a spanning tree of given maximum degree k. As to the other, we see a hamiltonian path as a path covering the vertex set of G and seek, in its absence, the minimum number of paths covering the vertex set of G. There are many papers dealing with the above two approaches of studying graphs that do not possess hamiltonian paths, but we will mention only a few of them. Most of the papers dealing with sufficient conditions for a graph to have a spanning tree with given maximum degree generalize a result formulated for hamiltonian paths. In [12] Win proved a generalization of Dirac's, see [4], classic sufficient condition for a graph to have a hamiltonian path. He showed that if G is connected and ${}_{v\in I} d(v) \geqslant n-1$ for each k-element independent set I, then G possesses a spanning tree T of the maximum degree at most k. The series of papers with that type of sufficient condition culminates in [9], where a generalization of a result due to Flandrin et al. [5] is proved. The structure of the spanning tree guaranteed by Win's result has been studied in [3]. As to the problem of covering vertices of a graph by paths, a result of Gallai and Milgram [6] states that vertices of G can be covered by at most $\alpha(G)$ vertex-disjoint paths. In fact the mentioned result is only a special case of their statement proved for directed graphs. Hoffman [8] showed that in case of $\kappa(G) < \alpha(G)$, the vertices of G can be covered by at most $\alpha(G) - \kappa(G)$ vertex-disjoint paths (recall that a result of Erdös

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and Chvátal [2] guaranties a hamiltonian cycle for $\kappa(G)\geqslant\alpha(G)$). Finally, Reed [11] proved that each cubic graph G can be covered by at most $\lceil \frac{n}{9} \rceil$ vertex-disjoint paths.

Our goal is to initiate a study of a problem that in a sense unifies the two approaches.

Problem 1. *Let G be a connected graph, and $k\geqslant 1$. Determine the minimum number S of trees $T_1, \dots, T_s$, $\Delta(T_i)\leqslant k$, $i=1,\dots,s$, covering all vertices of G.*

We will consider three modifications of the problem.

(1) T_i's are vertex-disjoint;
(2) T_i's are edge-disjoint;
(3) no condition on T_i's.

The answer to (1) is given by the following theorem.

Theorem 2. *Let $k\geqslant 1$. Then every connected graph G can be covered by at most* $\max(1, n-\delta k)$ *vertex-disjoint trees of maximum degree Δ, for $k\geqslant 2$, and by at most*$\max(\lceil \frac{n}{2} \rceil, n-\delta)$ *vertex-disjoint trees for $k=1$.*

The result is just a slight reformulation of a theorem in [1], see proof in Section 2. Moreover, the bound is the best possible as we need the number of trees given in the theorem to cover vertices of each graph H, $\delta(H)=\delta$, $H\subset K_\delta \vee \overline{K}_{n-\delta}$.

Clearly, for $k=1$, the vertex-disjoint and the edge-disjoint versions of the problem are equivalent. For $k\geqslant 2$, the edge-disjoint version of Problem 1 seems to be considerably more difficult than the vertex-disjoint one. We strongly believe that the following is true.

Conjecture 3. Let G be a connected graph, and $k\geqslant 2$. Then the vertices of G can be covered by $s\leqslant \left\lceil \frac{n-\delta}{\delta(k-1)+1} \right\rceil$ edge-disjoint trees of maximum degree $\leqslant k$.

The same graph as in case of Theorem 2 shows that if true, then the conjecture provides the best possible bound. We point out that the conjecture implies the existence of a spanning tree of maximum degree $\leqslant k$ for $\delta\geqslant \frac{n-1}{k}$. This is a sufficient condition for δ to imply Win's result, see above. As a further support for the conjecture we show that the statement is valid for $\delta=1$ and all $k\geqslant 2$, and for $k=2$ and all $\delta\geqslant 2$. That is, we prove

Theorem 4. *Let $k\in N$. Each connected graph of order n can be covered by $\left\lceil \frac{n-1}{k} \right\rceil$ edge-disjoint trees of maximum degree at most k.*

and

Theorem 5. *Each connected graph G can be covered by $\left\lceil \frac{n-\delta}{\delta+1} \right\rceil$ edge-disjoint paths.*

Finally, the last version of the problem, that is the case when the covering trees are allowed to share edges does not improve on the bound provided in the conjecture. It can be seen again by the same graph H as in Theorems 2 and 4.

The proofs of Theorems 2, 4, and 5 are given in Section 2. In Section 3 we briefly discuss the case of covering regular graphs by vertex-disjoint paths.

2. Proofs

Proof of Theorem 2. As mentioned in the introduction, the result follows immediately from the following theorem proved in [1].

Theorem CKR. *Let $k\geqslant 2$. Then every connected graph G contains a tree T of maximum degree at most k that either spans G or has order at least $k\delta(G)+1$.*

First consider the case $k \geqslant 2$. If G does not contain a spanning tree of maximum degree $\leqslant k$, then by Theorem CKR, at least $k\delta + 1$ vertices of G are covered by a tree T, $\Delta(T) \leqslant k$. The other $n - k\delta - 1$ vertices of G can be covered by singletons. Thus G can be covered by $n - k\delta - 1 + 1 = n - k\delta$ vertex-disjoint trees of maximum degree at most k.

Now let $k = 1$. By Theorem CKR applied for $k = 2$, G contains a path with $\min(n, 2\delta + 1)$ vertices. Therefore, G contains $\min(\left\lceil \frac{n-1}{2} \right\rceil, \delta)$ independent edges which cover $\min(2\left\lceil \frac{n-1}{2} \right\rceil, 2\delta)$ vertices of G. Covering by singletons the other vertices of G we get a covering by $\max(\left\lceil \frac{n}{2} \right\rceil, n - \delta)$ trees of maximum degree at most 1. □

Proof of Theorem 4. We start with a technical lemma.

Lemma 6. *Let $s, k, x_1, \ldots, x_s \in N$. Then*

$$\sum_{i=1}^{s} \left\lceil \frac{x_i}{k} \right\rceil - s + \left\lceil \frac{s}{k} \right\rceil \leqslant \left\lceil \frac{\sum_{i=1}^{s} x_i}{k} \right\rceil.$$

Proof of lemma. Let $x_i = q_i k + r_i$, $q_i \in N$, $0 \leqslant r_i < k$ for $i = 1, \ldots, s$. Assume $r_i = 0$ for $i = 1, \ldots, t$, and $r_i > 0$ for $i = t + 1, \ldots, s$. Then

$$\sum_{i=1}^{s} \left\lceil \frac{x_i}{k} \right\rceil - s + \left\lceil \frac{s}{k} \right\rceil = \sum_{i=1}^{s} q_i + (s - t) - s + \left\lceil \frac{s}{k} \right\rceil \leqslant \sum_{i=1}^{s} q_i + \left\lceil \frac{s-t}{k} \right\rceil$$
$$\leqslant \sum_{i=1}^{s} q_i + \left\lceil \sum_{i=t+1}^{s} \frac{r_i}{k} \right\rceil = \sum_{i=1}^{s} \frac{q_i k}{k} + \left\lceil \sum_{i=1}^{s} \frac{r_i}{k} \right\rceil = \left\lceil \sum_{i=1}^{s} \frac{q_i k + r_i}{k} \right\rceil = \left\lceil \frac{\sum_{i=1}^{s} x_i}{k} \right\rceil.$$

The proof of the lemma is complete. □

Let G be a connected graph of order n. We prove the statement by induction on n. Clearly the statement is true for $n \leqslant k + 1$. Now let $n > k + 1$. Suppose first that G is 2-connected. Then, by a result of [7] and also [10], there are in G disjoint sets of vertices A and B, $|A| = k$, $|B| = n - k$ such that both $[A]$ and $[B]$ are connected. By the induction hypothesis $[B]$, the subgraph induced by B, can be covered by $\left\lceil \frac{n-k-1}{k} \right\rceil = \left\lceil \frac{n-1}{k} \right\rceil - 1$ trees of maximum degree at most k. As $[A]$ can be covered by one such tree, the statement follows. So suppose now that G has a vertex v so that $G - v$ is disconnected. Denote by $C_1, \ldots, C_s$ the components of $G - v$. Further, remove from G some edges incident with v so that the resulting graph G' is connected and $d_{G'}(v) = s$. Thus, v is adjacent to exactly one vertex in C_i for $i = 1, \ldots, s$. By induction hypothesis, the subgraph G_i induced by vertices $v \cup V(C_i)$ can be covered by a set $\mathscr{T}_i$ of $\left\lceil \frac{|C_i|}{k} \right\rceil$ trees of maximum degree at most k. From each $\mathscr{T}_i$ we remove the tree T_i covering in G_i the vertex v. As v is a pendant vertex of G_i it is $d_{T_i}(v) = 1$. Group the s trees T_i into $\left\lceil \frac{s}{k} \right\rceil$ sets each comprising k trees T_i except possibly one set having less than k trees. In each set we amalgamate the trees of the set at the vertex v obtaining a new tree of maximum degree at most k. So, in total, we have a covering of G by $\sum_{i=1}^{s} \left\lceil \frac{|C_i|}{k} \right\rceil - s + \left\lceil \frac{s}{k} \right\rceil$ trees. By Lemma 6 $\sum_{i=1}^{s} \left\lceil \frac{|C_i|}{k} \right\rceil - s + \left\lceil \frac{s}{k} \right\rceil \leqslant \left\lceil \frac{\sum_{i=1}^{s} |C_i|}{k} \right\rceil = \left\lceil \frac{n-1}{k} \right\rceil$, and we are done.

Proof of Theorem 5. Let G be a counterexample to the statement. Set $\mathscr{P} = \{P_1, \ldots, P_t\}$, $t = \left\lceil \frac{n-\delta}{\delta+1} \right\rceil$, $|P_i| \geqslant |P_{i+1}|$, $i = 1, \ldots, t - 1$, to be a set of t edge-disjoint paths with the property:

(i) $\mathscr{P}$ covers the maximum possible number of vertices of G;
(ii) Subject to (i), $\sum_{i=1}^{t} |P_i|$ is minimum over all $\mathscr{P}$;
(iii) Subject to (ii), $\mathscr{P}$ is the "largest" element in the lexicographic order, that is, if $\mathscr{P}' = \{P'_1, \ldots, P'_t\}$ satisfies (ii), then there is an $j \in \{1, \ldots, t\}$ so that $|P_i| = |P'_i|$, $i < j$, and $|P_j| > P'_j$.

For each path P_i in $\mathscr{P}$ we choose one of its endvertices to be its initial vertex, and then we assign to P_i a set of vertices A_i as follows:

If $|P_i| = 1$, v being the single vertex of P_i, then $A_i = N(v) \cup v$. Otherwise, if $|P_i| > 1$, $P_i = x_1x_2 \ldots x_m$, x_1 being the initial vertex of P_i, then $x_1 \in A_i$. Further, let $y \in N(x_1)$. If $y \notin P_i$, then $y \in A_i$, if $y \in P_i$, then $y = x_j$, for some $j \in \{2, \ldots, m\}$, and we put $x_{j-1} \in A_i$, and also set $x_m \in A_i$. Finally, let e be a vertex not covered by $\mathscr{P}$, then we set $A_{t+1} = N(e) \cup e$.

By the definition of A_i we see that

$$|A_i| \geqslant \delta + 1. \tag{1}$$

To finish the proof it suffices to show that, for $1 \leqslant i < j \leqslant t + 1$,

$$A_i \cap A_j = \emptyset. \tag{2}$$

Indeed, with (2) in hands, and taking inequality (1) into account, we get $\left|\bigcup_{i=1}^{t+1} A_i\right| = \left(\left\lceil \frac{n-\delta}{\delta+1} \right\rceil + 1\right)(\delta + 1) \geqslant n - \delta + \delta + 1 > n$, a contradiction. Thus $\mathscr{P}$ covers all vertices of G.

Proof of (2). We start with some simple but useful properties of $\mathscr{P}$.

Claim 1. *If v is an endvertex of a path P_i then v is not covered by any path P_j, $j \neq i$. In particular, for no neighbor u of v, the edge uv belongs to a path P_j, $j \neq i$.*

To prove the claim it is sufficient to realize that if v were covered by another path of $\mathscr{P}$ then removal of v from P_i would result into a collection of paths contradicting condition (ii). As an consequence of Claim 1 we get:

Claim 2. *If v is an endvertex of a path P_i, then $v \notin A_j$ for $j \neq i$.*

If $v \in A_j$, $j \neq i$, then $vs \in E(G)$, where s is the initial vertex of P_j. However, then, for $|P_i| \geqslant |P_j|$, extending P_i to cover s and removing s from P_j (for $|P_i| < |P_j|$, removing v from P_i and extending P_j to cover v) provides a collection of edge-disjoint paths which satisfy both (i) and (ii) but precedes $\mathscr{P}$ in the lexicographic order, a contradiction.

Claim 3. *Let u be an internal vertex of P_i, and, at the same time, $u \in A_i$. Then u is not covered by any path P_j, $j \neq i$.*

Suppose by contradiction that u is covered by a path P_j, $j \neq i$. Let $P_i = v - aub - w$, with v being the initial vertex of P_i. Then $u \in A_i$ implies $vb \in E(G)$. However, then the collection of paths $\mathscr{P}' = \{P_1', \ldots, P_t'\}$, $P_k' = P_k$, $k \neq i$, $P_i' = a - vb - w$ contradicts (ii). Note, that Claim 1 guaranties that paths of $\mathscr{P}'$ are edge-disjoint as P_i' is edge-disjoint with other paths in $\mathscr{P}'$.

Claim 4. *For all $i = 1, \ldots, t$, the vertex $e \notin A_i$.*

To see this assume $e \in A_i$ for some i. Then the path P_i could be extended to cover e, which contradicts (i).

Now we are ready to prove the relation (2). Suppose to the contrary that $x \in A_i \cap A_j$. By Claim 4, $x \neq e$. We consider several cases. In each case we construct a collection $\mathscr{P}'$ of paths that leads to a contradiction with the choice of $\mathscr{P}$. For all $\mathscr{P}'$, the fact that $\mathscr{P}'$ consists of edge-disjoint paths follows from Claim 1.

(a) $x \notin P_i$. Let $P_i = v - w$, and let $x \notin P_j$. Assume $|P_i| \geqslant |P_j|$. Suppose first that $|P_j| = 1$, s being its single vertex. Then $xv, xs \in E(G)$, and $\mathscr{P}' = \{P_1', \ldots, P_t'\}$, $P_k' = P_k$, $k \notin \{i, j\}$, $P_i' = w - vxs$, $P_j' = e$, for $s \neq e$, $P_j' = \emptyset$, for $s = e$, contradicts (i). Suppose now that $|P_j| > 1$, $P_j = s - r$. Then we have $P_i \cap P_j \neq \emptyset$, otherwise $\mathscr{P}' = \{P_1', \ldots, P_t'\}$, $P_k' = P_k$, $k \notin \{i, j\}$, $P_i' = w - vxs - r$, $P_j' = e$, contradicts (i). Let $y \in P_i \cap P_j$ be such that $P_j = s - ayb - r$, and no vertex in $s - a$ belongs to $P_i \cap P_j$. By the Claim 1, $y \neq s$. Then $\mathscr{P}' = \{P_1', \ldots, P_t'\}$, $P_k' = P_k$, $k \notin \{i, j\}$, $P_i' = w - vxs - a$, $P_j' = b - r$ satisfies (i) and (ii) but $\mathscr{P}'$ is before $\mathscr{P}$ in the lexicographic order, a contradiction.

(b) $x \in P_i$. Then, by Claim 2, x is an internal vertex of P_i, say $P_i = v - axc - w$. Further, by Claim 3, $x \notin P_j$. By the definition of A_i, $vc \in E(G)$. Let $|P_j| > 1$, $P_j = sb - r$. If $|P_i| \geqslant |P_j|$, then the collection of paths $\mathscr{P}' = \{P_1', \ldots, P_t'\}$, $P_k' = P_k$, $k \notin \{i, j\}$, $P_i' = w - cv - axs$, $P_j' = b - r$ is before $\mathscr{P}$ in the lexicographic order. An obvious modification provides a proof for $|P_j| = 1$. For $|P_i| < |P_j|$, $\mathscr{P}' = \{P_1', \ldots, P_t'\}$, $P_k' = P_k$, $k \notin \{i, j\}$, $P_j' = r - sx$, $P_i' = a - vc - w$, is before $\mathscr{P}$ in the lexicographic order. So, in both cases we arrive at a contradiction.

The proof of (2), and therefore also of Theorem 5, is complete. □

3. Regular graphs

As mentioned in the introduction Reed proved that the vertices of each connected cubic graph G can be covered by at most $\lceil \frac{n}{9} \rceil$ vertex-disjoint-paths. The result is best possible. For other regular graphs we believe that:

Conjecture 7. The vertices of every connected d-regular graph can be covered by at most $\left\lceil \frac{n}{d+3+o(1)} \right\rceil$ vertex-disjoint paths.

If true, the bound is best possible. To see this, consider a tree T with a vertex v being its root. T contains only vertices of degree 1 and degree d, the root is of degree d, and all leafs of T are at the same distance from v. Further, let L_d be a graph which results from a complete graph K_{d+2}, by removing a matching M of size $\frac{d-3}{2}$ for d odd (of size $\frac{d-4}{2}$ for d even) and a path of length 2 for d odd (of length 3 for d even), that is vertex-disjoint with M. Thus, for d odd, all but one vertex, say x, and for d even, all but two vertices, say y, z, of L_d are of degree d. To obtain from T a regular graph G_d of degree d, we assign, for d odd, to each leaf w of T a copy of L_d, and identify the leaf w with the vertex x. For d even, we partition the leafs of T in pairs so that the distance of the leafs in the same group is at most 4 and the paths connecting the leafs belonging to the same pair are edge-disjoint. Then a copy of L_d is assigned to each pair of leafs. Finally, we identify one leaf from the pair with the vertex y the other with the vertex z. It is a matter of a routine calculation to verify that each covering of vertices of the graph G_d requires at least $\left\lceil \frac{n}{d+3+o(1)} \right\rceil$ vertex-disjoint paths. For $d = 3$, the above construction results in the same graph as given in [11] to show that the bound $\lceil \frac{n}{9} \rceil$ is the best possible.

References

[1] Y. Caro, I. Krasikov, Y. Roditty, On the largest tree of given maximum degree in a connected graph, J. Graph Theory 15 (1991) 7–13.
[2] V. Chvátal, P. Erdös, A note on hamiltonian circuits, Discrete Math. 2 (1972) 111–113.
[3] A. Czygrinov, G. Fan, G. Hurlbert, H.A. Kierstead, W.T. Trotter, Spanning trees of bounded degree, Electron. J. Combin. 8 (2001).
[4] G.A. Dirac, Some theorems on abstract graphs, Proc. London Math. Soc. 2 (1952) 69–81.
[5] E. Flandrin, H.A. Jung, H. Li, Hamiltonism, degree sum and neighborhood intersections, Discrete Math. 90 (1991) 41–52.
[6] T. Gallai, A.N. Milgram, Verallgemeinerung eines graphentheoretischen Satzen von Redei, Acta Sci. Math. (Szeged) 21 (1960) 181–186.
[7] E. Győri, On the division of graphs to connected subgraphs, Combinatorics (Proc. 1976), Colloq. Math. Soc. Janos Bolyai, No. 18, North-Holland, Amsterdam, 1978, pp. 485–496.
[8] A.J. Hoffman, Extending Greene's theorem to directed graphs, J. Combin. Theory Ser. A 34 (1983) 102–107.
[9] A. Kyaw, A sufficient condition for a graph to have a k-tree, Graphs and Combin. 17 (2001) 113–121.
[10] L. Lovász, A homology theory for spanning trees of a graph, Acta Math. Acad. Sci. Hungar. 30 (1977) 241–251.
[11] B. Reed, Paths, stars and the number three, Combin. Probl. Comput. 5 (1996) 277–295.
[12] S. Win, Existenz von Gerüsten mit vorgeschriebenem Maximalgrad in Graphen, Abhandlungen aus den matematischen Seminar der Universität Hamburg 43 (1975) 263–267.